On the Moments of the Multiple Correlation Coefficient in Samples from Normal Population

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Wishart (1930) has found the mean and the second moment coefficient of the multiple correlation coefficient in samples from a normal population when $b = 1, 2, 3$ and then assumed that the result is probably true without any restrictions as to whether $2b$ is even or odd. Also he has pointed out the difficulties in finding the mean and the standard deviation of the multiple correlation coefficient. Here I have found the m-th moment both of the square of the multiple correlation coefficient and also of the multiple correlation coefficient for unrestricted a, b. I have also found a method to find the mean and the second moment of the multiple correlation coefficient approximately.

Fisher's (1928) general distribution for the multiple correlation coefficient is

$$df = \frac{|\underline{a+b-1}}{|\underline{a-1}\,|\underline{b-1}}\,(1-\rho^2)^{a+b}\,F(a+b,\,a+b,\,a,\,\rho^2R^2)\times (R^2)^{a-1}\,(1-R^2)^{b-1}\,d\,(R^2).$$

Hence

$$\mu_m'\,(R^2) = \frac{|\underline{a+b-1}}{|\underline{a-1}\,|\underline{b-1}}\,(1-\rho^2)^{a+b}\sum_0^{a}\frac{(a+b)^2_s}{|\underline{s}\,(a)_s}\,\rho^{2s}\times \int_0^1 x^{a+s+m-1}\,(1-x)^{b-1}\,dx$$

$$= \frac{|\underline{a+b-1}}{|\underline{a-1}\,|\underline{b-1}}\cdot(1-\rho^2)^{a+b}\sum_0^{a}\frac{(a+b)^2_s}{|\underline{s}\,(a)_s}\,\rho^{2s}\times \frac{\Gamma(a+s+m)\,\Gamma(b)}{\Gamma(a+m+b+s)}\text{ where } (a)_s = a\,(a+1)\ldots(a+s-1)$$

and the term by term integration is permissible as the Hypergeometric function is uniformly convergent if $0 \leqslant \rho^2 < 1$.

From (11) and (12) we have

$$2 - 3x\,R(x) + (x^2 - 1)\,R^2(x) > 2 - 3x \cdot \frac{x^2 + 2}{x^3 + 3x}$$

$$+ (x^2 - 1) \cdot \frac{x}{x^2 + 1} \cdot \frac{x^3 + 5x}{x^3 + 6x^2 + 3}$$

which is easily seen to be > 0 for sufficiently large $x > 0$.
Hence

$$[\lambda(x) - x]\,[2\lambda(x) - x] - 1 > 0$$

which shows that the third moment of X, about its mean which is a measure of Skewness has the same sign as $E(X)$.

References

1. Birnbaum, Z. W. .. *Annals of Math. Statistics*, 1950, **21**, 272–79.
2. ———— .. *Ibid.*, 1942, **13**, 245–46.
3. Gordan, R. D. .. *Ibid.*, 1941, **12**, 364–66.
4. Kendall .. *Advanced Theory of Statistics*, **1**, 129–30.

Hence

$$\mu_m'(R^2) = \frac{(a+m-1)\ldots a}{(a+b+m-1)\ldots(a+b)}(1-\rho^2)^{a+b}$$

$$\times\ {}_3F_2(a+b, a+b, a+m, a, a+m+b, \rho^2)$$

Since

$$z^{\alpha_1+1}\ {}_3F_2(\alpha_1+1, \alpha_2, \alpha_3, \rho_1, \rho_2, z)$$

$$= \frac{z^2}{\alpha_1}\frac{d}{d_z}\ z^{\alpha_1}\ {}_3F_2(\alpha_1, \alpha_2, \alpha_3, \rho_1, \rho_2, z)$$

We have

$$\mu_m'(R^2) = \frac{(1-\rho^2)^{a+b}}{(a+b)\ldots(a+b+m+1)} \times$$

$$D^m F(a+b, a+b, a+m+b, \rho^2)$$

where

$$D = \frac{\rho^3}{2}\frac{\partial}{\partial\rho}.$$

Similarly we can prove that

$$\mu_m'(R) = \frac{\lfloor a+b-1}{\lfloor a-1}(1-\rho^2)^{a+b}\frac{\Gamma\left(a+\frac{m}{2}\right)}{\Gamma\left(a+b+\frac{m}{2}\right)} \times$$

$${}_3F_2\left(a+b, a+b, a+\frac{m}{2}, a, a+b+\tfrac{1}{2}, \rho^2\right)$$

$$= \frac{(1-\rho^2)^{a+b}\Gamma\left(a+\frac{m}{2}\right)}{\Gamma\left(a+b+\frac{m}{2}\right)} D^m F\left(a+b+\frac{m}{2}, a+b, \tfrac{1}{2}, \rho^2\right)$$

when b is integer.

Putting

$$m = 1$$

$$\mu_1'(R) = \overline{R} = \frac{\lfloor a+b-1\ \Gamma(a+\frac{1}{2})}{\lfloor a-1\ \Gamma\ a+b+\frac{1}{2}}(1-\rho^2)^{a+b} \times$$

$${}_3F_2(a+b, a+b, a+\tfrac{1}{2}, a, a+b+\tfrac{1}{2}, \rho^2)$$

when

$$\rho = 0,\ \overline{R} = \frac{\lfloor a+b-1\ \Gamma(a+\frac{1}{2})}{\lfloor a-1\ \Gamma(a+b+\frac{1}{2})}$$

which agrees with Wishart's result.

Example.—Let us consider the example considered by Pearson (1930). Let $\rho^2 = \cdot 5,\ a = 3,\ b = 47$.

$$\overline{R} = \frac{|49\, \Gamma 3 \cdot 5}{2\, \Gamma 50 \cdot 5} (\cdot 5)^{50}\ {}_3F_2\,(50,\ 50,\ 3 \cdot 5,\ 3,\ 50 \cdot 5,\ \cdot 5)$$

$$= \frac{|49\, \Gamma 3 \cdot 5}{2\, \Gamma 50 \cdot 5} (\cdot 5)^{50}\ {}_3F_2\,(50,\ 50,\ 3 \cdot 5,\ 3 \cdot 5,\ 51,\ \cdot 6)$$

$$= \frac{|49\, \Gamma 3 \cdot 5}{2\, \Gamma 50 \cdot 5} (\cdot 5)^{50}\ \mathrm{F}\,(50,\ 50,\ 51,\ \cdot 6) = \cdot 4 \text{ approximately.}$$

REFERENCES

Wishart, J. .. *Biometrica*, 1930, **22**, 224.

Fisher, R. A. .. *Proc. Roy. Soc.* **A**, 1928, **121**, 654.

Pearson, K. .. *Biometrica*, 1930, **22**, 362.